Myopic adaptation in games with strategic complementarities

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Abstract

In a finite game with strategic complementarities, every strategy profile is connected to a Nash equilibrium with a unilateral improvement path. If all but one players' strategies are scalar, every strategy profile is connected to a Nash equilibrium with a best-response improvement path. If all players have scalar strategies and each player is only affected by the sum of the partners' choices (in particular, if there are just two players), every best-response improvement path eventually leads to a Nash equilibrium.

Keywords: Strategic game; Individual improvement path; Best-response improvement path; Strategic complementarities

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1. Introduction

Individual myopic adaptation in strategic games has been studied since the time of A. Cournot. An important step was made recently by Monderer and Shapley (1996), who established a link between unilateral improvement dynamics and "potential functions" in finite games.

Milchtaich (1996) singled out three levels of nice behaviour of myopic adaptive processes in a finite game: (i) there is no infinite unilateral improvement path, i.e. every improvement path eventually leads to a Nash equilibrium (Monderer and Shapley's FIP

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property); (ii) there is no infinite best-response improvement path (FBRP); (iii) for every strategy profile there is a best-response improvement path leading from the profile to a Nash equilibrium (Young's, 1993, weak acyclicity). The last property ensures the convergence to an equilibrium with probability one of a best-response improvement process with a "sufficiently random" choice of an improvement at each stage. Young (1993) suggested a more complicated (and not quite myopic) stochastic scenario under which the property is also a sufficient condition for convergence. A fourth, weakest property should be added: (iv) for every strategy profile there is a one-sided improvement path leading from the profile to a Nash equilibrium; it ensures that such an adaptive process with a "sufficiently random" choice at each stage converges to an equilibrium with probability one. (It is unclear whether the property could be given an alternative interpretation in the style of Young, 1993.)

Kukushkin (1999, 2000) suggested to use the language of binary relations: both FIP and FBRP are easily restated as the acyclicity of appropriate individual domination relations. A potential is also understood as a strict order rather than a numeric function. Infinite games allow virtually the same treatment as finite ones if transfinite improvement paths are considered.

Voorneveld (2000) introduced the notion of a "best-response potential game," devoid of connections with any reasonable adaptive process, myopic or not. The property implies FBRP, but is not implied even by FIP, so it has no definite place in Milchtaich's classification. No natural class of games with this property was produced.

This letter addresses myopic adaptation processes in games with strategic complementarities (Topkis, 1979; Vives 1990; Milgrom and Roberts, 1990; Milgrom and Shannon, 1994). Topkis (1979, Algorithm I) proved the convergence of best-response improvement paths starting from the very bottom (or the very top) of the strategy profiles space. Vives (1990) extended the result to paths starting "very high" or "very low". Milgrom and Roberts (1990) considered rather general adaptive learning processes, but did not study their convergence. Here we are interested in convergence to an equilibrium of improvement paths with arbitrary starting points.

The main findings are these. In a finite game with strategic complementarities, every strategy profile is connected to a Nash equilibrium with a one-sided improvement path (Theorem 4). If the strategies of all players are scalar (which holds for all applications cited by Vives, 1990, or Milgrom and Roberts, 1990), then there is, at least, the weak acyclicity property (Theorem 3). If every player is only affected by the sum of the partners' choices (which holds for economic models such as the private provision of a public good or a public bad, or Cournot oligopoly), there is the FBRP property (Theorem 2); in the two-person case, the result is extended to infinite games (Theorem 1).

2. Basic definitions

A strategic game is defined by a finite set of players N and, for each $i \in N$, a set of strategies X_i and an ordinal utility function $u_i(x)$ defined on $X = \prod_{i \in N} X_i$. We always assume that each X_i is a compact metric space and each function u_i is, at least, upper semicontinuous in own variable x_i; then for each $i \in N$ and $x_{-i} \in X_{-i} = \prod_{j \neq i} X_j$, the best-response correspondence is well defined:

$$R_i(x_{-i}) = \{x_i \in X_i \mid \quad u_i(x_i, x_{-i}) = \max_{y_i \in X_i} u_i(y_i, x_{-i})\}.$$

We also define two types of individual improvement relations on X:

$$y \triangleright_i x \quad \Leftrightarrow \quad [y_{-i} = x_{-i} \;\&\; u_i(y) > u_i(x)], \quad y \triangleright x \quad \Leftrightarrow \quad \exists i \in N [y \triangleright_i x];$$

$$y \succ_i x \quad \Leftrightarrow \quad [y \triangleright_i x \;\&\; y_i \in R_i(x_{-i})], \quad y \succ x \quad \Leftrightarrow \quad \exists i \in N [y \succ_i x].$$

By definition, $x \in X$ is a Nash equilibrium if and only if x is a maximizer for the relation $\rhd$, i.e. if $y \rhd x$ for no $y \in X$. Under our topological assumptions, the same is true for $\succ$.

A Monderer-Shapley (M-S) path is a (finite or infinite) sequence $\{x^k\}_{k=0,1,\ldots}$, such that $x^{k+1} \rhd x^k$ whenever x^k and x^{k+1} are defined. With every M-S path $\{x^k\}_k$, a function $i(k)$ is associated, uniquely defined by $x^{k+1} \rhd_{i(k)} x^k$. An M-S cycle is an M-S path such that $x^0 = x^m$ for some m>0 (and x^k is defined just for $k = 0,1,\ldots m$). A Cournot path [cycle] is an M-S path [cycle] such that $x^{k+1} \succ x^k$ for all k.

From now on, we consider games where each X_i is a complete lattice and each utility function u_i has the properties of single crossing in (x_i, x_{-i}) (Milgrom and Shannon, 1994) and of pseudosupermodularity in x_i (Agliardi, 2000):

$$[y_i > x_i \ \& \ y_{-i} > x_{-i}] \Rightarrow [\text{sign}(u_i(y) - u_i(x_i, y_{-i})) \geq \text{sign}(u_i(y_i, x_{-i}) - u_i(x))] \quad \text{(SC)}$$

$$\begin{aligned} &\text{sign}(\max\{u_i(x_i \vee y_i, z_{-i}) - u_i(x_i, z_{-i}), u_i(x_i \vee y_i, z_{-i}) - u_i(y_i, z_{-i})\}) \geq \\ &\text{sign}(\max\{u_i(x_i, z_{-i}) - u_i(x_i \wedge y_i, z_{-i}), u_i(y_i, z_{-i}) - u_i(x_i \wedge y_i, z_{-i})\}) \end{aligned} \quad \text{(P)}$$

where $i \in N$, $x_i, y_i \in X_i$, $x_{-i}, y_{-i}, z_{-i} \in X_{-i}$, and sign($t$) is -1 if t<0, 0 if t=0 and 1 if t>0 (although subtraction is used in both definitions, the properties themselves are purely ordinal). (P) is satisfied automatically when X_i is a chain, which is quite often the case in the theorems below.

Lemma. If a game satisfies (SC) and (P), then, for each $i \in N$, $x_i, y_i \in X_i$, and $x_{-i}, y_{-i} \in X_{-i}$,

$$[y_{-i} \geq x_{-i} \ \& \ y_i \in R_i(y_{-i}) \ \& \ x_i \in R_i(x_{-i})] \quad \Rightarrow \quad [y_i \vee x_i \in R_i(y_{-i}) \ \& \ y_i \wedge x_i \in R_i(x_{-i})].$$

The statement means that $R_i(x_{-i})$ is a sublattice of X_i (pick $y_{-i} = x_{-i}$) and $R_i(\cdot)$ is increasing w.r.t. the strong set order defined by Veinott (see Topkis, 1979).

Indeed, $x_i \in R_i(x_{-i})$ implies $u_i(x) \geq u_i(x_i \wedge y_i, x_{-i})$, hence, by (P), $u_i(x_i \vee y_i, x_{-i}) \geq u_i(y_i, x_{-i})$, hence, by (SC), $u_i(x_i \vee y_i, y_{-i}) \geq u_i(y)$, hence $x_i \vee y_i \in R_i(y_{-i})$. On the other hand, $y_i \in R_i(y_{-i})$ implies that $u_i(y) \geq u_i(x_i \vee y_i, y_{-i})$, hence, by (SC), $u_i(y_i, x_{-i}) \geq u_i(x_i \vee y_i, x_{-i})$, hence, by (P), $u_i(x_i \wedge y_i, x_{-i}) \geq u_i(x)$, hence $x_i \wedge y_i \in R_i(x_{-i})$.

Remark. The lemma is obviously inspired by Proposition 3 of Agliardi (2000), but is formally independent of it.

In the following, a game satisfying both (SC) and (P) is called a game with strategic complementarities.

A game with additive aggregation (an AA game) is characterized by these properties: each X_i is a compact subset of the real line and $u_i(x) = U_i(x_i, \sum_{j \neq i} x_j)$, where U_i is defined on $X_i \times S_i$ and $S_i = \sum_{j \neq i} X_j$.

A game with additive single crossing (an ASC game) is an AA game such that, for each $i \in N$ and $x_i'' > x_i'$, the function $\text{sign}(U_i(x_i'', s_i) - U_i(x_i', s_i))$ increases (not necessarily strictly) in s_i. Obviously, an ASC game satisfies both (SC) and (P) above; however, if an AA game satisfies (SC), it need not be an ASC game unless there are just two players or the strategy sets are regular enough (e.g. all are closed intervals, or all are integer intervals). A stronger version of the Lemma holds for ASC games, where $x_{-i}, y_{-i} \in X_{-i}$ are replaced with $s_i', s_i \in S_i$, respectively.

3. Results

Theorem 1. Let there be a two-person game with strategic complementarities such that both strategy sets X_i are compact subsets of the real line and both best-response correspondences R_i are upper hemicontinuous. Then every infinite Cournot path converges to a Nash equilibrium.

Let there be an infinite Cournot path $\{x^k\}_{k=0,1,\ldots}$. Without restricting generality, $x^1 \succ_1 x^0$; hence $x^{2k+1} \succ_1 x^{2k}$ and $x^{2k+2} \succ_2 x^{2k+1}$ for all k. Since $x_2^1 = x_2^0$ and $x^2 \succ_2 x^1$, $x_2^2 = x_2^0$ is impossible; without restricting generality, we may assume that $x_2^2 > x_2^0$ (we could turn X_2 upside down if needed). Now we have $x^3 \succ_1 x^2$, hence $x_1^3 \in R_1(x_2^2)$ and $x_1^2 = x_1^1 \in R_1(x_2^0) \setminus R_1(x_2^2)$; therefore, by the Lemma, $x_1^3 > x_1^1$. Repeating the same reasoning inductively, we obtain $x_2^{2k+2} > x_2^{2k+1} = x_2^{2k}$ and $x_1^{2k+3} > x_1^{2k+2} = x_1^{2k+1}$ for all k=1,2,...

Now we may argue exactly as in Topkis (1979). Since each X_i is compact, there exist $x_i^\infty = \lim_{k\to\infty} x_i^k$ for both i=1,2. Moreover, $(x_1^{2k+1}, x_2^{2k}) \to (x_1^\infty, x_2^\infty)$ implies $x_1^\infty \in R_1(x_2^\infty)$, and $(x_1^{2k+1}, x_2^{2k+2}) \to (x_1^\infty, x_2^\infty)$ implies $x_2^\infty \in R_2(x_1^\infty)$. Therefore, (x_1^∞, x_2^∞) is a Nash equilibrium.

Theorem 1 obviously implies the absence of Cournot cycles, which, for a finite game, is equivalent to Milchtaich's FBRP. Generally, the absence of Cournot cycles does not even imply the existence of an equilibrium, see Example 2 of Kukushkin (1999).

Remark. By the well-known trick with reversing the order on one of the strategy sets, Theorem 1 can be applied to two-person games with strategic substitutes as well.

Theorem 2. A game with additive single crossing admits no Cournot cycle (hence in a finite ASC game every Cournot path, if continued while possible, leads to a Nash equilibrium).

Let $\Xi = \{x^0, \dots, x^{m-1}, x^m = x^0\}$ be a Cournot cycle in an ASC game. Pick x^k maximizing the sum $\sum_{i \in N} x_i^k$ and denote i=i(k), $s_i = \sum_{j \neq i} x_j^k$; there must be $x_i^k \notin R_i(s_i)$, $x_i^{k+1} \in R_i(s_i)$, and $x_i^{k+1} < x_i^k$. Since Ξ is a cycle, there must be a stage h such that i(h)=i and $x_i^{h+1} = x_i^k$; denote $s_i' = \sum_{j \neq i} x_j^h$ (then $x_i^k \in R_i(s_i')$) and consider three alternatives: If $s_i' > s_i$, then $\sum_{j \in N} x_j^{h+1} = x_i^k + s_i' > x_i^k + s_i = \sum_{j \in N} x_j^k$, contradicting the choice of x^k. If $s_i' = s_i$, then $x_i^k \in R_i(s_i)$ and $x_i^k \notin R_i(s_i)$ simultaneously. If $s_i' < s_i$, then $x_i^k \in R_i(s_i') \setminus R_i(s_i)$ and $x_i^{k+1} \in R_i(s_i)$ contradict the Lemma (for ASC games).

Remark. An exact analogue of the theorem is valid for games with additive strategic substitutes (cf. Kukushkin, 2000, Theorem 6.2); however, the proof is too complicated to be presented here.

Theorem 3. Let there be a finite game with strategic complementarities such that all but one players' strategy sets X_i, $i \neq 1$, are chains. Then, for every strategy profile $x \in X$, there is a (finite) Cournot path leading from x to a Nash equilibrium.

We denote, for every $x \in X$, $Y^{+}(x) = \{y \in X \mid y \succ x \ \& \ y > x\}$ and, for every $x_{-1} \in X_{-1}$, $r_1(x_{-1})$ the greatest element of $R_1(x_{-1})$. We call a Cournot path $\{x^k\}_k$ admissible if, whenever x^{k+1} is defined, the following requirements are satisfied:

(i) if $x_1^k \notin R_1(x_{-1}^k)$, then $i(k)$=1 and $x_1^{k+1} = r_1(x_{-1}^k)$;

(ii) if $x_1^k \in R_1(x_{-1}^k)$ and $Y^{+}(x^k) \neq \varnothing$, then $x^{k+1} \in Y^{+}(x^k)$.

Let us consider an arbitrary admissible Cournot path; requirement (i) allows us to assume, without restricting generality, that $x_1^0 \in R_1(x_{-1}^0)$. Suppose first that $Y^{+}(x^0) = \varnothing$; then $x_{i(0)}^1 < x_{i(0)}^0$. If there were $j \in N$ and $y \in Y^{+}(x^1)$ such that $y \succ_j x^1$, then, obviously, $j \neq i(0)$, hence $x_j^1 = x_j^0$ and $x_{-j}^1 < x_{-j}^0$; now the Lemma implies the existence of $z_j \in R_j(x_{-j}^0)$ such that $z_j \geq y_j$, hence $(z_j, x_{-j}^0) \in Y^{+}(x^0)$: a contradiction. An easy recursion using the monotonicity of r_1 shows that $Y^{+}(x^k) = \varnothing$ and $x^{k+1} < x^k$ for all k=0,1,..., hence a return to x^0 is impossible.

Now if $Y^{+}(x^0) \neq \varnothing$, then $x^1 > x^0$, and, again, an easy recursion shows that $x^{k+1} > x^k$ as long as $Y^{+}(x^k) \neq \varnothing$, so no cycling is possible here. Let m be the first stage at which $x_1^m \in R_1(x_{-1}^m)$ and $Y^{+}(x^m) = \varnothing$. Arguing exactly as in the previous paragraph, we see that the path cannot return to x^m.

Theorem 4. For every strategy profile $x \in X$ in every finite game with strategic complementarities, there is a (finite) M-S path leading from x to a Nash equilibrium.

We denote $Y^{+}(x)=\{y \in X \mid y \rhd x \;\&\; y>x\}$ and $Y^{-}(x)=\{y \in X \mid y \rhd x \;\&\; y<x\}$ for every $x \in X$, and call an M-S path $\{x^{k}\}_{k}$ admissible if, whenever x^{k+1} is defined, either $x^{k+1} \in Y^{+}(x^{k})$, or $Y^{+}(x^{k})=\varnothing$ and $x^{k+1} \in Y^{-}(x^{k})$. Arguing virtually in the same way as in the previous proof, we can see that an admissible cycle is impossible. Let us prove that an admissible path can only stop at a Nash eqilibrium.

Suppose the contrary: there is $x \in X$ such that both $Y^{+}(x)=\varnothing$ and $Y^{-}(x)=\varnothing$, but $u_i(y_i, x_{-i})>u_i(x)$ for some $y_i \in X_i$ (incomparable with x_i). We have $x_i \wedge y_i<x_i$, hence $u_i(x_i \wedge y_i, x_{-i}) \leq u_i(x)<u_i(y_i, x_{-i})$; similarly, $x_i \vee y_i>x_i$, hence $u_i(x_i \vee y_i, x_{-i}) \leq u_i(x)<u_i(y_i, x_{-i})$. Now we have a contradiction with (P).

Examples showing that the assumptions of the theorems are tight enough are available from the author. Here I can only list them: even a two-person game with strategic complementarities need not be weakly acyclic; a two-person game may admit a Cournot cycle even if the strategy set of one player is a chain; a three-person game may admit a Cournot cycle even if the strategy sets of all players are chains; a two-person game need not have the FIP property even if the strategy sets of both players are chains; even a two-person 2×2 game need not admit a Voorneveld potential.

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