ON MICROFOUNDATIONS OF THE DUAL THEORY OF CHOICE*

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Abstract

We show that Yaari's dual theory of choice under risk may be derived as an indirect utility when a risk-neutral agent faces financial imperfections. We consider an agent that maximizes expected discounted cash flows under a bid-ask spread in the credit market. It turns out that the agent evaluates lotteries as if she were maximizing Yaari's dual utility function. We also obtain representation results for the dual theory of choice for the case of unbounded lotteries.

Keywords: dual theory of choice, financial imperfections

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1 Introduction

The dual theory of choice under risk (DTC) was introduced in Yaari (1987) in order to resolve certain theoretical problems with expected utility. One of the most important properties of the latter is that risk aversion is equivalent to diminishing marginal utility. Yaari argued that risk aversion and diminishing marginal utility are 'horses of different colors' and put forward a theory which allowed for risk averse utility to be linear. The linearity property makes modelling behavior of risk-averse agents much simpler. This is why DTC has become a popular tool for testing robustness of various economic models that have been so far analyzed only in the expected utility framework. Demers and Demers (1990) apply DTC to firms' production decisions. Hadar and Seo (1995) examine portfolio choice and diversification in DTC. Doherty and Eeckhoudt (1995) study optimal insurance, Epstein and Zin (1990) use DTC as the simple utility with first-order risk aversion to offer an explanation of the equity premium. Volij (1999) studies revenue equivalence in auction with bidders that maximize DTC utility. Schmidt (1998) applies DTC to principal-agent problems.

Despite being a handy tool in theoretical models, the dual theory of choice has not done well in experimental studies. Both Harless and Camerer (1994) and Hey and Orme (1994) showed that DTC performs rather badly not only in absolute terms but also relative to other non-expected utility theories as well as relative to the expected utility theory.[1] This may not be surprising since the experiments were carried out on individuals. One should expect individuals to have diminishing marginal utility. On the other hand, firms and banks, especially in the long-run should have linear objective function. The firm's preference for net cash flows from a project should not depend upon those of other available projects. In the meanwhile, there exists a substantial empirical evidence that firms and even banks are risk-averse (see Rose (1989), Davidson et al. (1992), Park and Antonovitz (1992) etc.).

In the expected utility theory, risk-aversion of firms can be explained by market imperfections (Greenwald and Stiglitz (1990)). If credit markets are imperfect, a firm prefers certainty a mean-preserving spread of project payoffs decreases expected profits. Even if the firm maximizes net expected

[1] Though providing analysis of real economic choices rather than ones in artificially designed economic experiments, Zagonari (1995) tested only few theories.

cash flows, market imperfections may make it risk-averse.

The goal of our paper is to provide similar microfoundations for the dual theory of choice. We also consider a firm that is initially risk neutral. The firm evaluates lotteries (or project portfolios) according to the amount of expected discounted cash flows that can be obtained with those lotteries. We assume that the firm faces bid-ask spread in the financial market, i.e. the interest rate on loans is higher than the interest rate on deposits. The innovation of the paper is to assume that the firm anticipates the future need for borrowing if the returns are low and saving extra funds if the returns are high. Therefore firm can adjust its financial position before realization of stochastic payoffs. We derive indirect utility as a function of the distribution of returns and show that the indirect utility belongs to a certain subset of DTC utilities.

Another contribution of the paper is a generalization of DTC for unbounded random variables. Although infinite payments are not likely to occur in real world, in models they are quite common (e.g. normal distribution). We obtain representation for random variables with finite means. Our representation form turns out to be similar to one introduced in Roell (1987).

The paper is organized as follows. In Section 2 we introduce notation and obtain representation of DTC for unbounded lotteries. In Section 3, we show that an agent that faces a bid-ask spread in the financial market, evaluates lotteries as if she were a DTC utility maximizer. In Section 4 we check whether this result also holds for an arbitrary (not necessarily piecewise constant) interest rate schedule and show that it does not. Section 5 concludes. The Appendix contains proofs and technical material.

2 Dual theory of choice revisited

2.1 Notation

We shall consider a preference ordering $\succeq$ over real-valued random variables ('lotteries') X. We consider a probability space $(-\ ,\mathcal{A},\mathcal{P})$ where - is the set of states of nature, $\mathcal{A}$ is $\sigma-$algebra, $\mathcal{P}$ is a probability measure over $\mathcal{A}$. A random variable is a real-valued function of state of nature $X(\omega):-\ \longmapsto \Re$. For each random variable we introduce a cumulative distribution function

$F(x) = \text{Prob } \{X \leq x\}$. The distribution functions are right-continuous, non-decreasing and map $[-\infty, +\infty]$ onto $[0, 1]$. We will also use inverse distribution functions (IDF) $H(p) = F^{-1}(p) = \sup\{x : p \in \hat{F}(x)\}$, where $\hat{F}(x) = \{p : F(x - 0) \leq p \leq F(x)\}$ is the *set-valued distribution function.* According to this definition, a point (p, x) belongs to the graph of $H(p)$ if and only if the point (x, p) belongs to the graph of $F(x)$. $H(p)$ is a non-decreasing function that maps $[0, 1]$ onto $[-\infty, +\infty]$.

We will consider all random variables with finite expected value so that $|\int_{-\infty}^{+\infty} x dF(x)| = |\int_0^1 H(p)dp| < \infty$. Then $H(p) \in L^1(0, 1)$ and $F(x) \in L^1(-\infty, +\infty)$. Denote M a set of all non-decreasing upper semi-continuous functions that belong to $L^1(0, 1)$.[2] Then M includes IDF of all random variables with finite means. M is a semi-linear space. Indeed, for any $H_1, H_2 \in M$ and $\alpha \geq 0$ we have $H_1 + H_2 \in M$ and $\alpha H_1 \in M$. There is a zero element in M: $H_0(p) \equiv 0$ for all $p \in [0, 1]$. Indeed, $H_0 = 0 \cdot H$ and $H_0 + H = H$ for any $H \in M$.

We do not require $H(0)$ or $H(1)$ to be finite and therefore allow for unbounded random variables. The L^1 norm is very convenient for dealing with IDFs: the distance between two distribution functions $\int_{-\infty}^{+\infty} |F_1(x) - F_2(x)|dX$ in terms of space $L^1(-\infty, +\infty)$ coincides with distance between corresponding IDFs $\int_0^1 |H_1(p) - H_2(p)|dp$ in terms of $L^1(0, 1)$.

We shall also need a definition of comotonicity.

Definition 1 *Two random variables X and Y are said to be comonotonic if $(X(\omega) - Y(\omega))(X(\omega') - Y(\omega')) \geq 0$ holds for any pair of states of nature ω, ω'.*

2.2 Dual theory of choice for unbounded lotteries

We will use Yaari's axiom set: take the first four axioms of expected utility theory and replace the Neuman-Morgenstern independence axiom with Yaari's dual independence axiom.

Axiom 1 *Neutrality (A1). If inverse distribution functions of two random variables X and Y coincide $H_X(p) = H_Y(p)$ then $X \sim Y$.*

[2] Strictly speaking, L^1 is a space of *classes* of functions which may differ on a set of measure zero. According to our definition of IDF, we take the representative of this class which is upper semi-continuous or, which is the same, right semi-continuous. However, the choice of a particular representative function within the class does not matter.

This means that instead of dealing with random variables we can simply define preference ordering on M. By definition, $H_X(p) = H_Y(p)$ is equivalent to $F_X(p) = F_Y(p)$. Hereinafter, we shall understand $H_X(p) = H_Y(p)$ as equality almost everywhere (i.e. maybe except for a set of measure zero).

Axiom 2 *Continuity (A2). The preference ordering is continuous on M in terms of L^1 norm i.e. if $X \succ Y$ then there exists $\varepsilon > 0$ such that $\|H_Z - H_Y\| < \varepsilon$ implies $X \succ Z$.*

Axiom 3 *Monotonicity (A3). If $H_X(p) \geq H_Y(p)$ then $X \succeq Y$.*

Axiom 4 *Certainty equivalence (A4). There exists a functional U that assigns a real number ('certainty equivalent') to any random variable (with a finite mean) such that $X \succeq Y$ if and only if $U(X) \geq U(Y)$. For any real a holds $U(\xi_a) = a$, where ξ_a is a degenerate random variable which takes value a with probability 1.*

Axiom 5 *Linearity with regard to comonotonic random variables (A5). Suppose that A4 is satisfied. If random variables X and Y are comonotonic, $U(X + Y) = U(X) + U(Y)$, where $U(\cdot)$ is the certainty equivalence functional.*

The monotonicity axiom A3 is equivalent to conventional monotonicity in terms of first-order stochastic dominance. The fourth axiom usually states that the preference is a complete weak order. We replace this axiom with the certainty equivalence axiom A4.[3] A5 is equivalent to Yaari's dual independence axiom and characterizes a class of utilities which can be nicely represented and coincides with Yaari's dual utility functions in case of uniformly bounded support of random variables. If instead of A4 we introduced a complete weak order axiom, then A5 could have been reformulated as follows: $X \succeq Z$ implies $X + Y \succeq Z + Y$ for any comonotonic X, Y and Z.

The intuition for axiom A5 is as follows. If random variables X and Y cannot be used as a hedge against each other, the agent should be prepared to pay for $X + Y$ as much as what she is willing to pay for X plus what she is willing to pay for Y. The 'no-hedge' condition is precisely what the Definition 1 implies: increase in X can never be offset by decrease in Y.

[3] There is a large literature on cardinal vs. ordinal utility (see a review in Fishburn (1994)) and it is not our goal to discuss the issue here.

Apparently, if A1,A2, A4 hold, A5 implies that for any random variable X and any real $\alpha \geq 0$ $U(\alpha X) = \alpha U(X)$.

Axioms A1-A4 allow to define the utility functional $U(\cdot)$ as a monotonic continuous functional on the set of IDFs M. The Yaari's dual independence axiom A5 simply states that this functional is linear.

Proposition 1 *Suppose A1-A4 hold. Then utility $U(X)$ satisfies A5 if and only if it is a linear continuous functional on inverse distribution functions normalized to certainty equivalence i.e. for all inverse distribution functions H_1, H_2, $H_3 \in M$ and non-negative real α:*

- $H_2 = \alpha H_1$ implies that $U(Y) = \alpha U(X)$ for all random variables X and Y such that $H_X = H_1$, $H_Y = H_2$.
- $H_3 = H_1 + H_2$ implies that $U(Z) = U(X) + U(Y)$ for all random variables X, Y, Z such that $H_X = H_1$, $H_Y = H_2$, $H_Z = H_3$.
- If $H(p) \equiv 1$ for all p then $U(X) = 1$ for any random variable X such that $H_X = H$.

We could have also replaced 'for any non-negative α' with 'for any real α' and the statement would still be the same because there is no negative α such that there exist X, Y: $H_Y = \alpha H_X$ (both H_Y and H_X must be non-decreasing).

Theorem 1 *Utility satisfies axioms A1-A5 if and only if it can be represented in the following form:*

$$U(H) = \int_0^1 h(p)H(p)dp \tag{1}$$

where the generating function $h(p) \in L^\infty[0,1]$ is non-negative $h(p) \geq 0$ and normalized to one $\int_0^1 h(p)dp = 1$.

The idea of the proof provided in the Appendix is to apply a well-known result of functional analysis that the space of linear continuous functionals is isomorphic to the conjugate space.[4]

[4]Yaari's proof is different. Its idea is to 'lay Neuman-Morgenstern result on its side' (Yaari (1987)).

Remark: If $F(x)$ is continuous then utility may also be represented in terms of cumulative distribution function

$$U(F) = \int_{-\infty}^{+\infty} h(F(x))x dF(x). \tag{2}$$

Note that by construction this representation gives utility which is properly defined for all random variables with finite expected value including the unbounded ones. In the meantime, Yaari's representation

$$U = \int_{-\infty}^{+\infty} g(F(x))dx \tag{3}$$

is defined only for random variables that are uniformly bounded (at least from one end). Indeed, if X is unbounded then

$$\int_{-\infty}^{+\infty} g(F(x))dx = \int_{-\infty}^{0} g(F(x))dx + \int_{0}^{+\infty} g(F(x))dx$$

is finite only if $g(0) = g(1) = 0$. But monotonicity axiom requires that g is non-decreasing. Thus the only functional (3) defined for unbounded random variables is the trivial one $g \equiv 0$.

For uniformly bounded random variables, though, integration by parts can be used to convert (2) to Yaari's form and back (which has been done in Roell (1987) and Demers and Demers (1990)).

The generating functions $h(p)$ must belong to $L^{\infty}(0,1)$ because we allow for all random variables with finite means. If we allowed only for random variables with both finite mean and variance we would have $h(p) \in L^2(0,1)$ instead ($L^2(0,1)$ is the conjugate space for $L^2(0,1)$).

2.3 Risk-aversion

Similarly to Yaari's characterization of risk aversion in DTC (Yaari, 1986 and 1987), we shall determine conditions on $h(\cdot)$ for utility (1) to be risk-averse.

In order to define risk aversion, we shall use Rotschild-Stiglitz concept of mean-preserving spread. Consider arbitrary (with finite expected value) random variable X and some uncorrelated noise ξ ($E(\xi|X) = 0$). Then $X + \xi$ is a mean-preserving spread of X and therefore risk-averse agents should prefer X to $X + \xi$.

Definition 2 *Utility $U(\cdot)$ is said to be risk-averse if $U(X) \geq U(X + \xi)$ for all X and ξ such that $E(\xi|X) = 0$.*

It is well known that if neutrality axiom A1 holds, such definition of risk-aversion can be re-written in terms of distribution functions. Utility functional is *risk averse* if for any random variables X and Y with the same mean $EX = EY$ the following is true: if inequality

$$\int_{-\infty}^{a} F_X(x)dx \leq \int_{-\infty}^{a} F_Y(y)dy \tag{4}$$

holds for any real a then $U(X) \geq U(Y)$.[5]

One can easily reformulate risk aversion in terms of IDF. The (4) is equivalent to

$$\|H_X(p) - a\| \leq \|H_Y(p) - a\|. \tag{5}$$

Theorem 2 *The utility functional (1) is risk averse if and only if the generating function $h(p)$ is non-increasing.*

3 Microfoundations of dual theory of choice

3.1 The basic model

In this Subsection we will consider a simple two-period model of a risk-neutral agent that faces a bid-ask spread in the credit market. Namely, the interest rate the agent pays on her loans is R_l while she can only save at the risk-free rate $R_s < R_l$. The agent is neutral to risk and maximizes expected discounted withdrawn earnings in periods 0 and 1 with the discount rate Δ. Therefore the agent has constant marginal utility so that the model below is applicable rather to firms than to households.

In order to make the model non-trivial, we assume (all rates are gross)

$$R_s < \Delta < R_l \ . \tag{6}$$

The agent is to evaluate a lottery (e.g. a risky project) that pays X in the period 1. The distribution function of payoffs $F(\cdot)$ is such that the expected value of X is finite: $|EX| = \left|\int_{-\infty}^{+\infty} XdF(X)\right| < \infty$.

[5] As shown in Rotchild and Stiglitz (1970) for such X and Y there exists a noise ξ that meets conditions of Definition 2 so that Y and $X + \xi$ have the same distribution.

Agent chooses to withdraw C in the first period. If $X > C$ then the agent saves $S = X - C$ and therefore the second-period payoff will be $R_s(X - C)$. If $X < C$ then the agent will have to borrow $L = C - X$ and will have to pay $R_l(C - X)$ in the second period. The expected present discounted value of the project is

$$U = \int_{-\infty}^{+\infty} dF(x) \left\{ C + \Delta^{-1} \left(R_s [x - C]_+ - R_l [C - x]_+ \right) \right\} \qquad (7)$$

We shall calculate the agent's evaluation of the project U as a function of $F(\cdot)$ for three scenarios of agent's choice of C. First, let us consider the case of extreme flexibility where the first period spending decision C is taken after the realization of X is observed (timeline (a) in Fig. 1). The solution is straightforward: take $C(X) = X$. Then $U = EX$ — agent remains risk-neutral.

The opposite extreme is to assume that the agent cannot vary C at all (timeline (b) in Fig. 1). Suppose that $C = C^*$ is exogenously set by someone else (or by the agent herself but before she even learns $F(\cdot)$). The resulting functional U is an expected utility one similar to those discussed in Greenwald and Stiglitz (1990), Eeckhoudt et al. (1997). Indeed, agent gets expected value of a piecewise-linear utility function with a kink at $X = C^*$. Eeckhoudt et al. (1997) show that this functional has a first-order risk aversion.

The innovation of our paper is to study the intermediate case where the agent can vary C but has to take this decision before X is observed (timeline (c) in Fig. 1). In this case, the agent's evaluation of the lottery is

$$U = \max_C \left\{ C - \frac{R_l}{\Delta} \int_{-\infty}^{C} (C - x) dF(x) + \frac{R_s}{\Delta} \int_{C}^{\infty} (x - C) dF(x) \right\}. \qquad (8)$$

Proposition 2 *The solution C to maximization problem (8) satisfies*[6]

$$F(C - 0) \leq (\Delta - R_s)/(R_l - R_s) \leq F(C) \; . \qquad (9)$$

If $F(X)$ is continuous, (9) takes the form

$$F(C) = (\Delta - R_s)/(R_l - R_s) \qquad (10)$$

[6]Hereinafter $F(x - 0) = \lim_{x' \to x, x' < x} F(x')$.

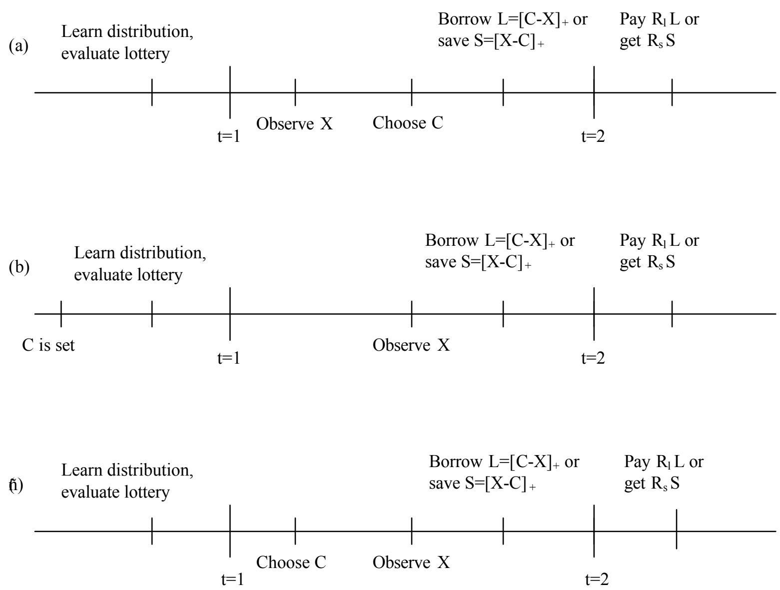

Figure 1: Agent's evaluation of a lottery X depends on timeline. Under timeline (a) the agent chooses first-period withdrawals C *after* observing realization of X. In this case she is risk-neutral and prefers a lottery with higher EX. In the case (b) where the agent evaluates the lottery after C is set, her preferences are described by a risk-averse expected utility. If the agent chooses first-period withdrawals C *before* knowing realization of X (case (c)) she is risk-averse and maximizes a Yaari utility (8).

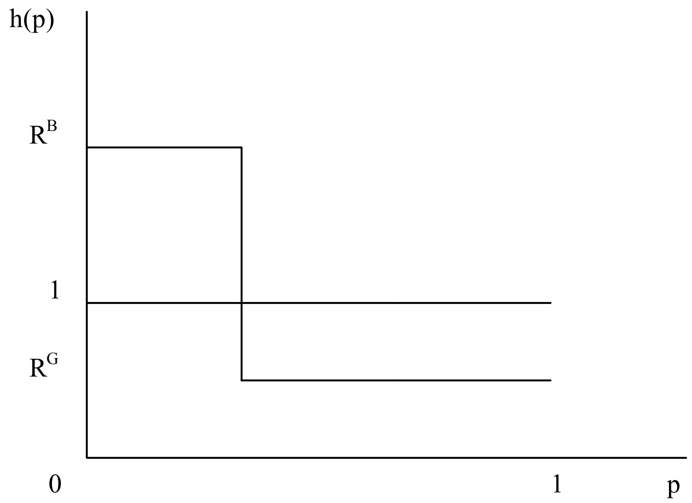

Figure 2: The generating function (11) for the simplest DTC utility.

Formula (10) implies $C = H(p^*)$, where $p^* = (\Delta - R_s)/(R_l - R_s)$. The first-period spending is simply the value of inverse distribution function of the project's payoff taken at a given point $(\Delta - R_s)/(R_l - R_s)$. Therefore if project's payoff increases by a dollar in all states of nature, C also goes up by one dollar. If payoffs in all states double, C doubles as well. If there are two comonotonic projects, the corresponding values of C add up. Thus, C as a functional of project satisfies the dual independence axiom A5.

It turns out that overall evaluation of the project $U(\cdot)$ is also a Yaari functional.

Proposition 3 *Formula (8) defines a DTC utility functional (1) with the generating function*

$$h(p) = \begin{cases} R^B \text{ if } p < p^* \\ R^G \text{ if } p \geq p^* \end{cases} \tag{11}$$

(see Fig.2), where $R^G = R_s/\Delta$, $R^B = R_s/\Delta$, $p^* = (1 - R^G)/(R^B - R^G)$.

Thus, (8) provides the simplest possible risk-averse DTC functional. The agent calculates a 'weighted average' of payoffs with higher weights R^B for bad states of nature ($p < (1 - R^G)/(R^B - R^G)$) and lower weights R^G for good states of nature.

Apparently, the converse statement is also true. For an arbitrary risk-averse DTC utility functional (1) such that $h(p)$ takes only two values, there exist such R_l and R_s that this functional can be derived as the value function in (8). This class of utilities is the set of simplest possible risk-averse rank-dependent utilities that are defined as follows: the states are ranked in the order of payoffs and divided into the set of the 'good' ones and that of the 'bad' ones. The agent's payoff of getting a dollar in a good state is R^G, while that of getting a dollar in a bad state is $R^B < R^G$. Such utilities are fully characterized by two parameters. The first one is the cutoff p^* between the 'good' states of nature and the 'bad' ones. The other parameter is the relative difference of utility of getting a dollar in a good state and that of getting a dollar in a bad state $r = R^G/R^B$. Using normalization $1 = \int_0^1 h(p)dp = R^B p^* + (1 - p^*)R^G$ we obtain (11).

Even these simplest DTC utilities may be consistent with Allais paradox (see Appendix B). In order to explain Allais paradox in the original Allais' formulation, R^B should be very high relative to R^G: $R^B/R^G > 40$. This means that in terms of the simplest DTC utility, Allais-consistent agents expect very high punishment for lack of cash compared with very a small reward for excess cash.

Besides the capital market imperfections, the analysis above can also be applied to an agent operating under a piece-wise linear tax schedule. Consider an agent that faces positive tax rate for positive net cash flows and zero tax for negative cash flows ('no-loss-offset' rule). The model will be absolutely identical to the one above. The discontinuity in the tax rate makes effective R^G lower: $R^G = (1-\tau)R^B$, where τ is the tax rate for positive profits.

3.2 The dynamic extension

In this Subsection we will provide an infinite horizon extension of the basic model of Subsection 3.1. Suppose that the agent receives uncertain payoffs X_t at each period t. The payoffs X_t are independent identically distributed random variables with distribution function $F(\cdot)$ known to the agent. The expected value of X is finite: $\left|\int_{-\infty}^{+\infty} X dF(X)\right| < \infty$.

At each period t, the agent can hold non-interest bearing cash M_t and invest in risk-free bonds S_t with gross interest rate R_s. The agent can also borrow L_t at a gross interest rate R_l. The bonds have one-period maturity and the loans must be repaid next period, too. In addition to (6), we assume that the return on bonds is greater than one on money

$$R_s > 1. \tag{12}$$

The agent maximizes expected discounted cash flows C_t :

$$V = E \sum_{t=0}^{\infty} \Delta^{-t} C_t \ , \tag{13}$$

by choosing $M_{t+1} \geq 0$, $L_{t+1} \geq 0$, $S_{t+1} \geq 0$ and C_t under the following constraint

$$M_{t+1} = M_t - R_l L_t + R_s S_t + L_{t+1} - S_{t+1} + X_t - C_t \tag{14}$$

and initial conditions

$$M_0 = M \geq 0, \quad L_0 = L \geq 0, \quad S_0 = S \geq 0. \tag{15}$$

There is also a no-Ponzi-game condition

$$\text{Prob} \left\{ \overline{\lim_{t \to \infty}} \Delta^{-t} L_t = 0 \right\} = 1 \ . \tag{16}$$

The agent chooses withdrawals C_t before realization of X_t and therefore can only rely upon information on state variables M_t, L_t, S_t. In the meanwhile, the choice of M_{t+1}, L_{t+1}, S_{t+1} is made after revelation of X_t and may also depend upon X_t. We do not impose non-negativity constraint on C_t. If the agent is in a hard situation (low M_t, S_t, low expected value of X_t or high indebtedness L_t), then she has to plan losses (or try to attract additional capital e.g. via issue of new equity or sales of fixed assets).

Proposition 4 *If (6), (12) hold, the stochastic programming problem (13)-(16) has a solution. The value of the objective function (13) as a function of the initial condition satisfies the Bellman equation*

$$V(M,L,S) = \max_{C} \left\{ C + \Delta^{-1} E_X \max_{M',L',S' \geq 0} V(M',L',S') \right\} \qquad (17)$$

where the inside maximum is taken subject to $M' = M - R_l L + R_s S + L' - S' + X - C$ and the random variable X has a c.d.f. $F(\cdot)$.

Proposition 5 *For both stochastic programming problem (13)-(16) and dynamic programming problem (17) there exists the same unique solution which is as follows. The Bellman function is:*

$$V(M,L,S) = M - R_l L + R_s S + U\Delta/(\Delta - 1) \qquad (18)$$

where U is given by (8). The control variables are

$$M' = 0,\; L' = [\Phi - X]_+,\; S' = [X - \Phi]_+,\; C = \Phi + M - R_l L + R_s S, \qquad (19)$$

where Φ is determined from condition

$$F(\Phi - 0) \leq (\Delta - R_s)/(R_l - R_s) \leq F(\Phi)\ . \qquad (20)$$

If $F(X)$ is continuous, (20) is re-written in the form

$$F(\Phi) = (\Delta - R_s)/(R_l - R_s). \qquad (21)$$

Thus, the Bellman function in a model with infinite horizon is the sum of combination of state variables $M - R_l L + R_s S$ and the present value of getting DTC utility (8) every period ad infinitum.

4 Evaluation of lotteries under non-linear financial contracts

A natural question emerges how broad is the class of DTC functionals that can be derived as a indirect utility of an agent that faces financial imperfections. Formally, we can ask whether some other DTC utilities can be derived if we introduce arbitrary schedules of interest rates as functions of amount invested/borrowed. Suppose that the gross interest payments are $I(\xi)\Delta$ where ξ is the net savings (negative if borrowings), Δ is the gross discount rate and $I(\xi)$ is the interest rate schedule normalized by discount factor. In the previous Subsection we considered the simplest case of bid-ask spread

$$I(\xi) = \begin{cases} R^G\xi \text{ if } \xi \geq 0 \\ R^B\xi \text{ if } \xi < 0 \end{cases}. \tag{22}$$

In real life, firms (as well as households) face interest rates that not only differ for savings and loans but also depend on amounts saved or borrowed. Let us extend our analysis for the case of arbitrary $I(\xi)$. The expected discounted earnings are $U = C + \int_{-\infty}^{+\infty} I(x - C)dF(x)$. Again, the timeline is crucial. If the agent chooses C knowing X (similar to (a) in Fig. 1) then the agent remains risk-neutral $U = EX + \max_\xi \{I(\xi) - \xi\}$. If C cannot be varied at all (case (b) in Fig. 1), then agent evaluates the projects according to an expected utility functional. The agent maximizes expectation of non-linear utility function $I(x-C)$. Apparently, such agent is risk-averse whenever $I(\xi)$ is concave.

If agent chooses C before observing X then the expected discounted earnings are

$$U = \max_C \left\{ C + \int_0^1 I(H(p) - C)dp \right\}, \tag{23}$$

where $H(\cdot)$ is the IDF of X. We already know that for a particular case (22) with $R^G < 1 < R^B$ the utility (23) is a Yaari one with the generating function (11). The problem that we will address now is whether there are some other interest rate schedules that generate Yaari's utility. The answer to this question is negative. It turns out that no other DTC functional can be derived as an indirect utility in the maximization problem (23).

Theorem 3 *Let $I(\xi)$ be continuous and differentiable on $(-\infty, +\infty)$ except maybe a countable set. Let us also assume that there exist (finite or infinite) limits $R_+ = \lim_{\xi\to\infty} I(\xi)/\xi$ and $R_- = \lim_{\xi\to-\infty} I(\xi)/\xi$. Then utility (23) is a DTC one (1) if and only if $R_+ \leq 1 \leq R_-$ and there exists a real number a such that*

$$I(\xi) = \begin{cases} a + R_+(\xi - a) \text{ if } \xi \geq a \\ a + R_-(\xi - a) \text{ if } \xi < a \end{cases}. \tag{24}$$

The Theorem complements the results obtained in the Subsection 3.1, where we showed that all risk-averse DTC utilities (1) with a generating function that takes only two values R^B and R^G can be derived as preferences of a risk-neutral agent that faces a bid-ask spread in the interest rates. Theorem 3 essentially says that no other DTC utility can be derived in such a model. No interest rate schedule that satisfies the technical conditions of the Theorem can result in a DTC utility except the two-rate interest schedule (24) which is essentially equivalent to the bid-ask spread we have already studied. Indeed, substituting (24) into (23) we obtain the same DTC utility as in the Subsection 3.1 with $R^G = R_+$ and $R^B = R_-$.

It is also worth emphasizing that only risk-averse utilities have microfoundations. The DTC utilities with the generating function (11) and $R^G > R^B$ cannot be derived in the form (8). Whenever $R_+ > R_-$, there is no finite solution to the maximization problem in (23).[7]

5 Conclusions

The main contribution of the paper is to show that some dual theory of choice utilities can be obtained as an indirect utility of a risk-neutral agent that faces a bid-ask spread in the credit market. These utilities are the simplest risk-averse rank-dependent utilities which are parameterized by two numbers: the cutoff point that separates the 'good' and the 'bad' states of nature and the relative penalty for being in a bad state. The model is simple: the agent has to take the decision on the first-period withdrawals before she learns realization of stochastic returns. If she withdraws too much today, she would have to borrow which will result in high interest payments and therefore low consumption tomorrow. If she withdraws too little, she would have to save and get returns tomorrow at a relatively low deposit rate. It is the agent's ability to adjust the first-period withdrawal knowing the distribution of payoffs but not the actual realization that makes her evaluation of the lottery a risk-averse dual utility functional. If the agent were not able to adjust, her preferences would be described by a risk-averse expected utility. Vice versa, if the agent were able to adjust the withdrawals after observing the realization, she would remain risk-neutral.

[7]The risk-loving utility functionals with $R^G > R^B$ are similar to the Pangloss value functional in Krugman (1998).

We also show that no DTC utility outside this particular class has such microfoundations even if we allow for arbitrary non-linear financial contracts. Only simplest risk-averse rank-dependent utilities can be obtained as a solution in an optimization problem. This result reveals a striking difference between expected utility and dual theory of choice. Any expected utility can be derived as a preference ordering of an agent under financial imperfections in the model where agent cannot vary the first-period consumption. In the world where agent adjusts her first-period consumption, things are very different. Under arbitrary non-linear interest schedules, the agent still has a constant absolute risk aversion, but not necessarily a constant relative risk aversion. It turns out that the only interest rate schedule that results in a DTC utility is the bid-ask spread studied above where the agent pays constant interest rate on loans and gets a constant though lower rate on deposits.

Another contribution of the paper is generalization of DTC for unbounded lotteries with finite means. The original Yaari's formulation cannot be extended to the unbounded case. The representation form that we get is rather similar to that of Roell (1987) though the latter was also obtained for uniformly bounded lotteries.

We believe that our results justify the growing popularity of DTC as an appropriate (and very simple) tool for analysis of firms' decision-making under risk. Being tested on individuals, DTC has performed rather poorly. However, it has not yet been empirically tested on firms or banks. There is some evidence that firms' and banks' are risk-averse. Rose (1989), Davidson et al. (1992), Park and Antonovitz (1992) and many other authors prove that firms indeed seek insurance and diversification. On the other hand, only risk-averse expected utility has been tested on firms; it remains unknown whether DTC would perform better or worse in such tests.

APPENDICES

APPENDIX A. Proofs.

Proof of Proposition 1. First if $Y = \alpha X$ then $H_Y = \alpha H_X$ and if X and Y are comonotonic then $H_{X+Y} = H_X + H_Y$. So whenever utility is linear with regard to IDF it satisfies A5.
To prove the 'only if' part we need to use neutrality axiom A1. Consider arbitrary H_1, $H_2 \in M$, $H_2 = \alpha H_1$. Take any random variable X such that $H_X = H_1$ and then consider random variable αX. By definition $H_2 = H_{\alpha X}$. Then for any random variable Y such that $H_Y = H_2 = H_{\alpha X}$ neutrality requires $U(Y) = U(\alpha X) = \alpha U(X)$.
Similarly, consider $H_1, H_2, H_3 \in M$: $H_3 = H_1 + H_2$. Then take any Z: $H_Z = H_3$ and define X and Y in the following way: for every real $z \in [H_3(0), H_3(1)]$ find $\hat{p}(z)$ such that $H_3(\hat{p}) = z$. Then X takes value $H_1(\hat{p}(z))$ and Y takes value $H_2(\hat{p}(z))$ whenever Z takes value z. Thus we have obtained comonotonic X, Y such that $H_X = H_1$, $H_Y = H_2$ and $X + Y = Z$. For them $U(X+Y) = U(Z) = U(X) + U(Y)$. Due to A1, this will also be true for all X, Y, Z such that $H_X = H_1$, $H_Y = H_2$, $H_Z = H_3$. ■

Proof of Theorem 1. The utility functional U is defined and linear on M. Let us extend it onto the whole space $L^1[0,1]$. For any function $g \in L^1(0,1)$ there exists a representation $g = \lim_{n\to\infty} g_n$, where $g_n = H_n^1 - H_n^2$, and $H_n^1, H_n^2 \in M$. Indeed, the set of continuous functions is dense in $L^1(0,1)$; the set of polynomials is dense in the set of continuous functions, and every polynomial can be represented as a difference of two monotonic functions.

Define $U(g) = \lim_{n\to\infty} U(H_n^1) - U(H_n^2)$. Due to linearity of U over M there is no ambiguity: $U(g)$ does not depend upon representation of g. Let $g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} \tilde{g}_n$, where $g_n = H_n^1 - H_n^2$, and $\tilde{g}_n = \tilde{H}_n^1 - \tilde{H}_n^2$. Then $\|g_n - \tilde{g}_n\| = \|H_n^1 - H_n^2 - \tilde{H}_n^1 + \tilde{H}_n^2\| \to 0$. Grouping the terms and using continuity of U over M we obtain $U(H_n^1 + \tilde{H}_n^2) - U(\tilde{H}_n^1 + H_n^2) \to 0$. Then applying linearity of U on M and re-grouping terms back we get $(U(H_n^1) - U(H_n^2)) - (U(\tilde{H}_n^1) - U(\tilde{H}_n^2)) \to 0$.
Thus, every A5 utility generates a linear continuous functional on $L^1[0,1]$. It is well known (see Yosida (1978)) that the set of all such functionals is isomorphic to conjugate space $L^\infty[0,1]$ i.e. each utility functional can be represented as $\int_0^1 h(p)H(p)dp$ where $h \in L^\infty[0,1]$.

We have already used A1, A2 and A5. Axioms A3 and A4 reduce the set of possible generation functions $h(\cdot)$ further. Monotonicity implies $h(p) \geq 0$ and certainty equivalence axiom requires normalization $\int_0^1 h(p)dp = 1$. ■

Proof of Theorem 2. The proof basically follows Yaari (1987). First let us consider two random variables with IDF $H_1(p)$ and $H_2(p)$ such that $\int_0^1 H_1(p)dp = \int_0^1 H_2(p)dp$ and for any a $\|H_1(p)-a\| \leq \|H_2(p)-a\|$. Consider $H(p) = H_1(p) - H_2(p)$. Let us now prove that $I(q) = \int_0^q H(p)dp \geq 0$ for any $q \in [0,1]$.

Then let us divide $[0,1]$ into intervals (p_k, p_{k+1}) on which $H(p)$ has same sign ($H(p)$ changes sign at least once). Then for every sign change point p_k there exist a_k such that $H_{1,2}(p) \geq a_k$ whenever $p > p_k$ and $H_{1,2}(p) \leq a_k$ whenever $p < p_k$. Then using the risk aversion condition we obtain $0 \geq \|H_1(p)-a_k\| - \|H_2(p)-a_k\| = -2\left(\int_0^{p_k}((H_1(p)-a_k)-(H_2(p)-a_k))\,dp\right)$, i.e. $I(p_k) \geq 0$. Similarly, $I(p_{k+1}) \geq 0$. Furthermore, for an arbitrary $p \in (p_k, p_{k+1})$ we have either $I(p_k) \leq I(p) \leq I(p_{k+1})$ (if $H(p)$ is non-negative on (p_k, p_{k+1})) or $I(p_{k+1}) \leq I(p) \leq I(p_k)$ (if $H(p)$ is non-positive on (p_k, p_{k+1})).

Now when we proved that $I(q) \geq 0$ for all $q \in [0,1]$, we may integrate by parts: $U(H_1) - U(H_2) = \int_0^1 h(p)H(p)dp = \int_0^1 h(p)dI(p) = h(p)I(p)|_0^1 - \int_0^1 I(p)dh(p) \geq 0$.

The non-integral term is zero since $I(0) = I(1) = 0$. The integral term is non-positive (since $h(p)$ is monotonic, $dh(p)$ is a non-positive measure and since $I(p)$ is an absolutely continuous function, the integral exists and is non-positive).

Let us now assume that there is a set $P \subset [0,1]$ of non-zero measure on which a risk averse $h(p)$ is strictly increasing. Let us divide the set into two subsets of equal measure: $P = P_1 \oplus P_2$, $\int_{P_1} dp = \int_{P_2} dp$, such that $p_1 \leq p_2$ for all $p_1 \in P_1$, $p_2 \in P_2$. Consider two random variables ξ and η: $H_\xi(p) = H_\eta(p)$ for all $p \notin P$, $H_\xi(p) = 1/2$ for $p \in P$ and $H_\eta(p) = 0$ if $p \in P_1$ and $H_\eta(p) = 1$ if $p_2 \in P_2$. Then $\|H_\xi(p) - a\| \leq \|H_\eta(p) - a\|$ for any real a. Risk aversion implies that $U(\xi) \geq U(\eta)$. Hence $\frac{1}{2}\int_P h(p)dp \geq \int_{P_2} h(p)dp$, or $\int_{P_1} h(p)dp \geq \int_{P_2} h(p)dp$. This means that $h(p)$ can not be strictly increasing on P. Proved by contradiction. ■

Proof of Proposition 4. The partial sum of (13) is as follows.

$$\sum_{t=\tau}^{T} \Delta^{\tau-t} C_t = \{M_\tau - R_l L_\tau + R_s S_\tau\} + \sum_{t=\tau}^{T} \Delta^{\tau-t} X_t -$$

$$-(\Delta-1)\sum_{t=\tau+1}^{T}\Delta^{\tau-t}M_t-(R_l-\Delta)\sum_{t=\tau+1}^{T}\Delta^{\tau-t}L_t-$$
$$-(\Delta-R_s)\sum_{t=\tau+1}^{T}\Delta^{\tau-t}S_t+\Delta^{\tau-T}L_{T+1}-$$
$$-\Delta^{\tau-T}\left(M_{T+1}+S_{T+1}\right)$$

Under assumptions about interest rates (6) and no-Ponzi-game condition (16)expected value of $\sum_{t=\tau}^{T}\Delta^{\tau-t}C_{t+1}$ is bounded from above by $M_t-R_lL_t+R_sS_t+const$. Therefore the problem has a solution and the Bellman function is well-defined. Let us define a state $Z_t=\{M_t,L_t,S_t\}$. Then the agent's strategy is a quadruple of Borel functions - $=\{\tilde{M}(t,Z,X),\ \tilde{L}(t,Z,X),\ \tilde{S}(t,Z,X),\ \tilde{C}(t,Z)\}$, where $\tilde{M}(t,Z,X)$, $\tilde{L}(t,Z,X)$, $\tilde{S}(t,Z,X)$ are non-negative and for all M,L,S and X holds $\tilde{M}(t,\{M,L,S\},X)=M+X-R_lL+R_sS+\tilde{L}(t,\{M,L,S\},X)-\tilde{S}(t,\{M,L,S\},X)-\tilde{C}(t,\{M,L,S\})$. The strategy defines a Markov process $Z_{t+1}=\{\tilde{M}(t,Z_t,X_t),\tilde{L}(t,Z_t,X_t),\tilde{S}(t,Z_t,X_t)\}=G\ \ (t,Z_t,X_t)$ and a series of random variables $C_t=\tilde{C}(t,Z_t)$. Therefore for each strategy - and initial conditions Z one can calculate the expected net present value $V\ \ (\tau,Z)=E\{\sum_{t=\tau}^{T}\Delta^{\tau-t}C_t|Z_\tau=Z\}$. We have just shown that V is bounded and the maximization problem max $J\ \ (\tau,Z)$ is well-defined. The Kolmogorov equation for $V\ \ (t,Z)$ is as follows:

$$V\ \ (t,Z)=\tilde{C}(t,Z)+\Delta^{-1}J\ \ (t+1,G\ \ (t,Z,X)). \tag{25}$$

By definition the optimal strategy -̂ satisfies $V_{\hat{}}(t,Z)\geq V\ \ (t,Z)$ for all t,Z and - . Therefore it also satisfies the Bellman equation

$$J\ \ (t,Z)=\sup_{(t,\cdot)}\tilde{C}(t,Z)+\Delta^{-1}V\ \ (t+1,G\ \ (t,Z,X)). \tag{26}$$

Apparently, time index t can be omitted. Substituting for Z and - , we obtain (17). ■

Proof of Proposition 5. In order to solve the Bellman equation, we shall introduce $\Phi=C-(M-R_lL+R_sS)$. The Bellman equation can then be re-written as

$$V(M,L,S) = M - R_l L + R_s S + \max_{\Phi} \left\{ \Phi + \Delta^{-1} E_X \max_{\substack{M',L',S' \geq 0 \\ M' = L' - S' + X - \Phi}} V(M',L',S') \right\}. \tag{27}$$

The Bellman function is therefore linear:

$$V(M,L,S) = M - R_l L + R_s S + (\Delta - 1)^{-1} \Delta U, \tag{28}$$

where

$$U = \max_{\Phi} \left\{ \Phi + \Delta^{-1} E_X \max_{\substack{L',S' \geq 0 \\ L' - S' + X - \Phi \geq 0}} L' - S' + X - \Phi - R_l L' + R_s S' \right\}. \tag{29}$$

According to (6), the expression $L' - S' + R_l L' + R_s S'$ increases whenever L' and S' decrease by the same amount. Therefore $L' = [\Phi - X]_+$ and $S' = [X - \Phi]_+$. Then

$$U = \max_{\Phi} \left\{ \Phi - R^B \int_{-\infty}^{\Phi} (\Phi - x) dF(x) + R^G \int_{\Phi}^{\infty} (x - \Phi) dF(x) \right\}, \tag{30}$$

where $R^B = R_l/\Delta > 1$ and $R^G = R_s/\Delta < 1$. The first- and second-order conditions imply (20). ■

Proof of Proposition 3. First, let us prove that the maximization problem (23) has a finite solution only if $R_+ \leq 1 \leq R_-$. Indeed, if $R_+ > 1$, the agent would choose $C = \infty$ and would get an infinite utility, while if $R_- < 1$, infinite utility is achieved by taking $C = -\infty$.

All DTC utilities (1) have both constant absolute and constant relative risk-aversion. We shall prove now that in order for (23) having a constant absolute and constant relative risk-aversion, the interest payment schedule $I(\xi)$ must be represented in the form (24).

It is clear that (23) has constant absolute risk aversion: for any real b

$$U(H+b) = b + \max_C \left\{ C - b + \int_0^1 I(H(p) - (C-b))dp \right\} = b + U(H). \quad (31)$$

Let us now check whether (23) has constant relative risk aversion. For any real $\alpha > 0$ we should have $\alpha^{-1}U(\alpha H) = U(H)$. Apparently,

$$\alpha^{-1}U(\alpha H) = \max_C \left\{ C + \int_0^1 \alpha^{-1} I(\alpha(H(p) - C))dp \right\}. \quad (32)$$

First, we shall consider $I(\xi)$ such that for any real non-negative α holds $I(\alpha\xi) = \alpha I(\xi)$. This implies

$$I(\xi) = \begin{cases} R_+\xi \text{ if } \xi \geq 0 \\ R_-\xi \text{ if } \xi < 0 \end{cases}.$$

If $R_+ < R_-$ then it is the DTC utility derived in the Subsection 3.1 (it is a particular case of (24) for $a = 0$). If $R_+ = R_- = 1$ then it is the case of risk-neutrality.

Now we shall see what happens if for some α and ξ we have $I(\alpha\xi) \neq \alpha I(\xi)$. Take this α and introduce $J(\xi) = \alpha^{-1}I(\alpha\xi)$. Then (32) takes the form

$$\alpha^{-1}U(\alpha H) = \max_C \left\{ C + \int_0^1 J(H(p) - C)dp \right\}. \quad (33)$$

Let $C_I^*(H)$ and $C_J^*(H)$ be solutions of the maximization problems in (23) and (33), correspondingly. Since $\alpha^{-1}U(\alpha H) = U(H)$, we have

$$C_J^*(H) + \int_0^1 J(H(p) - C_J^*(H))dp = C_I^*(H) + \int_0^1 I(H(p) - C_I^*(H))dp.$$

This equality should hold for any H. Consider $\tilde{H}$ sufficiently close to H. According to the envelope theorem,

$$\begin{aligned} U(\tilde{H}) - U(H) &= \int_0^1 I'(H(p) - C_I^*(H))\delta(p)dp + O(||\delta||^2) \\ \alpha^{-1}U(\alpha\tilde{H}) - \alpha^{-1}U(\alpha H) &= \int_0^1 J'(H(p) - C_J^*(H))\delta(p)dp + O(||\delta||^2) \end{aligned}$$

where $\delta = \tilde{H} - H$ is small. Hence, $\int_0^1 J'(H(p) - C_J^*(H))\delta(p)dp = \int_0^1 I'(H(p) - C_I^*(H))\delta(p)dp + O(||\delta||^2)$. Since the set of all possible deviations $\delta(p)$ is sufficiently rich we should have $J'(\xi) = I'(\xi + a)$ almost everywhere. Here $a = C_J^*(H) - C_I^*(H)$.

The condition $J'(\xi) = I'(\xi + a)$ implies $J(\xi) = J(0) + I(\xi + a) - I(a)$. Substituting into $\alpha^{-1}U(\alpha H) = U(H)$ we obtain $J(0) = I(a) - a$. Therefore $J(\xi) = I(\xi + a) - a$.

Thus, utility (23) has constant relative risk aversion only if for every $\alpha > 0$ there exists a real a such that $J(\xi) = \alpha^{-1}I(\alpha\xi) = I(\xi + a) - a$ for all ξ.

Let us see what happens if $\alpha \to \infty$. For a given $\eta > 0$

$$\eta \lim_{\alpha\xi \to +\infty} \frac{I(\alpha\eta)}{\alpha\eta} = I(\eta + a) - a \tag{34}$$

Thus, $I(\eta + a) = a + R_+\eta$. Similarly, for $\eta < 0$ we have $I(\eta + a) = a + R_-\eta$. Substituting $\eta + a$ for ξ we obtain the formula (24).

We have proved that utility (23) can be a DTC one only if the interest payment schedule is (24). Let us check whether the condition is also (24) sufficient. Let us take arbitrary a and $R_+ \leq 1 \leq R_-$. Then making straightforward calculations one can show that the utility (23) is the one introduced in the Subsection 3.1, i.e. the utility (1) with the generating function (11) where $R^G = R_+$ and $R^B = R_-$. ■

APPENDIX B. Allais paradox

Consider the Allais paradox described in Allais (1979). There are four lotteries. A is getting \$1 mil. with probability 1. B is getting \$1 mil. with probability 0.89, \$5 mil. with probability 0.1 and \$0 mil. with probability 0.01. C is getting \$1 mil. with probability 0.11 and \$0 mil. with probability 0.89. D is getting \$5 mil. with probability 0.1 and \$0 mil. with probability 0.9. The commonly observed preference ($A \succ B$ and $D \succ C$) cannot be explained by EU theory. We will show that the simplest DTC utility (1),(11) explains Allais paradox at certain values of parameters R^G and R^B. Indeed,

$$\begin{aligned}
&U(A) = 1, \\
&U(B) = (0.89 + 0.5)R^G + (R^B - R^G)\left([p^* - 0.01]_+ + 4[p^* - 0.9]_+\right), \\
&U(C) = 0.11R^G + (R^B - R^G)[p^* - 0.89]_+, \\
&U(D) = 0.5R^G + 5(R^B - R^G)[p^* - 0.9]_+,
\end{aligned}$$

where $p^* = (1 - R^G)/(R^B - R^G)$.

Straightforward calculations prove that $U(A) > U(B)$ and $U(C) > U(D)$ if and only if simultaneously $R^G < 1/1.39$, $0.89R^B + 0.5R^G > 1$, $R^B > 40R^G$. Such R^B and R^G apparently exist.

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